R-CLOSEDNESS AND UPPER SEMICONTINUITY

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Abstract. Let $\mathcal{F}$ be a pointwise almost periodic decomposition of a compact metrizable space X. Then we show that $\mathcal{F}$ is R-closed if and only if $\hat{\mathcal{F}}$ is usc. On the other hand, let G be a flow on a compact metrizable space and H a finite index normal subgroup. Then G is R-closed if and only if so is H. Moreover, if there is a finite index normal subgroup H of an R-closed flow G on a compact manifold such that the orbit closures of H consist of codimension k compact connected submanifolds and "few singularities" for $k = 1$ or 2, then the orbit class space of G is a compact k-dimensional manifold with conners. In addition, let v be a nontrivial R-closed vector field on a connected compact 3-manifold M. Then one of the following holds: 1) The orbit class space $M/\hat{v}$ is $[0, 1]$ or S^1 and each interior point of $M/\hat{v}$ is two dimensional. 2) $\mathrm{Per}(v)$ is open dense and $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$. 3) There is a nontrivial non-toral minimal set.

1. Preliminaries

In [ES], they have show that if a continuous mapping f of a topological space X in itself is either pointwise recurrent or pointwise almost periodic then so is f^k for each positive integer k. This results is extended into flow cases (see Theorem 2.24, 4.04, and 7.04 [GH]). In [Y3], one have shown the analogous result for R-closed homeomorphisms. In this paper, we extend into the R-closed flow cases.

The leaf space of a compact continuous codimension two foliation $\mathcal{F}$ of a compact manifold M is a compact orbifold [E], [EMS], [E2], [Vo], [Vo3]. On the other hand, there are non-R-closed compact foliations and non R-closed flows each of whose orbits is compact for codimension $q > 2$ [S], [EV], [Vo2]. In [Y2], the author has shown that the set of R-closed decompositions on compact manifolds contains properly the set of codimension one or two foliations which is minimal or compact. In this paper, for $k = 1$ or 2, we show that the quotient space of a codimension k compact connected decomposition with "few singularities" defined by a flow is a compact k-dimensional manifold. In addition, let v a nontrivial R-closed vector field on a connected compact 3-manifold. Then one of the following holds: 1) The orbit class space $M/\hat{v}$ is $[0, 1]$ or S^1 and each interior point of $M/\hat{v}$ is two dimensional. 2) $\mathrm{Per}(v)$ is open dense and $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$. 3) There is a nontrivial non-toral minimal set.

By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a set X such that $X = \sqcup \mathcal{F}$. Let $\mathcal{F}$ be a decomposition of a topological space X. For any $x \in X$, denote by L_x or $\mathcal{F}(x)$ the element of $\mathcal{F}$ containing x. For a subset $A \subseteq X$, A is saturated if $A = \sqcup_{x \in A} L_x$. $\mathcal{F}$ is upper semicontinuous (usc) if

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each element of $\mathcal{F}$ is both closed and compact and, for any $L \in \mathcal{F}$ and for any open neighbourhood U of L, there is a saturated neighbourhood of L contained in U. Note that we can choose the above U open. $\mathcal{F}$ is R-closed if $R := \{(x, y) \mid y \in \overline{L_x}\}$ is closed. $\mathcal{F}$ is pointwise almost periodic if the set of all closures of elements of $\mathcal{F}$ also is a decomposition. Then denote by $\hat{\mathcal{F}}$ the decomposition of closures and by $M/\hat{\mathcal{F}}$ the quotient space, called the orbit class space. By a flow, we mean a continuous action of a topological group G on a topological space X. For a flow G, denote by $\mathcal{F}_G$ the set of orbits of G. Recall a flow G is R-closed if the set $\mathcal{F}_G$ of orbits is an R-closed decomposition. Then G is R-closed if and only if $R := \{(x, y) \mid y \in \overline{G(x)}\}$ is closed. Recall that each R-closed decomposition is pointwise almost periodic. For an R-closed vector field v, write $M/\hat{v} := M/\hat{\mathcal{F}}_v$.

2. R-closedness and Upper semi continuity

Now we show the following key lemma.

Lemma 2.1. *Let $\mathcal{F}$ be a decomposition of a Hausdorff space X. If $\mathcal{F}$ is pointwise almost periodic and $\hat{\mathcal{F}}$ is usc, then $\mathcal{F}$ is R-closed. If X is compact metrizable and $\mathcal{F}$ is R-closed, then $\mathcal{F}$ is pointwise almost periodic and $\hat{\mathcal{F}}$ is usc.*

Proof. Suppose that $\mathcal{F}$ is pointwise almost periodic and that $\hat{\mathcal{F}}$ is usc. By Proposition 1.2.1 [D2] (p.13), we have that $X/\mathcal{F}$ is Hausdorff. By Lemma 2.3 [Y], we obtain that $\mathcal{F}$ is R-closed. Conversely, suppose that X is compact metrizable and $\mathcal{F}$ is R-closed. By Lemma 1.6 [Y], we have that $\mathcal{F}$ is pointwise almost periodic and that the quotient map $p : X \to X/\hat{\mathcal{F}}$ is closed. Since X is compact Hausdorff, we obtain that each element of $\hat{\mathcal{F}}$ is compact. By Proposition 1.1.1 [D2](p.8), we have that $\mathcal{F}$ is usc. □

Lemma 2.3 [Y] implies the following result.

Proposition 2.2. *Let $\mathcal{F}$ be a pointwise almost periodic decomposition of a compact metrizable space X. The following are equivalent:*
1) $\mathcal{F}$ is R-closed.
2) $\hat{\mathcal{F}}$ is usc.
3) $X/\hat{\mathcal{F}}$ is Hausdorff.

3. Inherited properties of R-closed flows

Recall that a subgroup H of a topological group G is syndetic if there is a compact subset of G such that $K \cdot H = G$.

Lemma 3.1. *Let G be a flow on a topological space X and H a syndetic subgroup of G. If H is R-closed, then so is G. Moreover $\overline{G \cdot x} = K \cdot \overline{H \cdot x}$ for any $x \in Y$ where K is a compact subgroup with $K \cdot H = G$.*

Proof. For any flow $\pi : G \times Y \to Y$ on a topological space Y, we claim that $K \cdot C$ is closed for a closed subset C of Y. Indeed, fix a point $x \in Y - K \cdot C$. Then $Y - C$ is an open neighborhood of $K^{-1} \cdot x$ and so $\pi^{-1}(Y - C)$ is an open neighborhood of $K^{-1} \times \{x\}$. Since $K^{-1} \times \{x\}$ is compact, by the tube theorem, there is an open neighborhood U of x such that $K^{-1} \times U \subseteq \pi^{-1}(Y - C)$. Then $K^{-1} \cdot U \subseteq Y - C$. Therefore $(K^{-1} \cdot U) \cap C = \emptyset$ and so $U \cap (K \cdot C) = \emptyset$. This shows that $K \cdot C$ is closed. Let $R_G := \{(x, y) \mid y \in \overline{G \cdot x}\}$ and $R_H := \{(x, y) \mid y \in \overline{H \cdot x}\}$. Suppose that H is R-closed. Then $K \cdot \overline{H \cdot x} \subseteq \overline{G \cdot x} = \overline{K \cdot (H \cdot x)} \subseteq \overline{K \cdot \overline{H \cdot x}}$. By the above

claim, we have $\overline{G \cdot x} = \overline{(K \cdot H) \cdot x} = \overline{K \cdot \overline{H \cdot x}} = K \cdot \overline{H \cdot x}$. Consider an action of G on $X \times X$ by $g \cdot (x, y) := (x, g^{-1} \cdot y)$. Since K is compact and R_H is closed, the above claim implies that $R_G = K \cdot R_H$ is closed. $\square$

For a subset V, write $\hat{\mathcal{F}}(V) := \mathrm{Sat}_{\hat{\mathcal{F}}}(V) = \cup_{x \in V} \hat{\mathcal{F}}(x)$. We generalize Lemma 1.1 [Y3] to flows. This statement is an analogous result for recurrence and pointwise almost periodicity (see Theorem 2.24, 4.04, and 7.04 [GH]).

Lemma 3.2. *Let G a flow on a compact metrizable space X and H a finite index normal subgroup. Then G is R-closed if and only if so is H.*

Proof. By Lemma 3.1, the R-closedness of H implies one of G. Conversely, suppose that G is R-closed. Let n be the index of H and $\{f_1, \dots, f_{n-1}\}$ a subset of G such that $G = H \sqcup H f_1 \sqcup \cdots \sqcup H f_{n-1}$. Put $\mathcal{F} := \mathcal{F}_G$. By Corollary 1.4 [Y], we have that G is pointwise almost periodic. By Theorem 2.24 [GH], we have that H is also pointwise almost periodic. By Proposition 2.2, $\hat{\mathcal{F}}$ is usc and it suffices to show that $\hat{\mathcal{F}}_H$ is usc. Note that $\mathcal{F}_H(x) \subseteq \mathcal{F}(x)$ and so $\hat{\mathcal{F}}_H(x) \subseteq \hat{\mathcal{F}}(x)$. For $x \in X$ with $\hat{\mathcal{F}}_H(x) = \hat{\mathcal{F}}(x)$ and for any open neighbourhood U of $\hat{\mathcal{F}}(x) = \hat{\mathcal{F}}_H(x)$, since $\hat{\mathcal{F}}$ is usc, there is a $\hat{\mathcal{F}}$-saturated open neighbourhood V of $\hat{\mathcal{F}}(x)$ contained in U. Since $\hat{\mathcal{F}}_H(x) \subseteq \hat{\mathcal{F}}(x)$, we have that V is also a $\hat{\mathcal{F}}_H$-saturated open neighbourhood V of $\hat{\mathcal{F}}(x)$. Fix any $x \in X$ with $\hat{\mathcal{F}}(x) \neq \hat{\mathcal{F}}_H(x)$. Put $\hat{L}_1 := \hat{\mathcal{F}}_H(x)$ and $\{\hat{L}_2, \dots, \hat{L}_k\} := \{\hat{\mathcal{F}}(f_l(x)) \mid l = 1, \dots, n-1\}$ such that $\hat{L}_i \cap \hat{L}_j = \emptyset$ for any $i \neq j \in \{1, \dots, k\}$. Let $\hat{L}' = \hat{L}_2 \sqcup \cdots \sqcup \hat{L}_k$. Then $\hat{L}_1$ and $\hat{L}'$ are closed and $\hat{\mathcal{F}}(x) = \hat{L}_1 \sqcup \cdots \sqcup \hat{L}_k = \hat{L}_1 \sqcup \hat{L}'$. For any sufficiently small $\varepsilon > 0$, let $U_{1,\varepsilon} = B_\varepsilon(\hat{L}_1)$ (resp. $U'_\varepsilon = B_\varepsilon(\hat{L}')$) be the open ε-ball of $\hat{L}_1$ (resp. $\hat{L}'$). Then $\overline{U_{1,\varepsilon/2}} \subseteq U_{1,\varepsilon}$ and $\overline{U'_{\varepsilon/2}} \subseteq U'_\varepsilon$. Since ε is small and X is normal, we obtain $U_{1,\varepsilon} \cap U'_\varepsilon = \emptyset$. Since $\hat{\mathcal{F}}$ is usc, there are neighbourhoods $V_{1,\varepsilon} \subseteq U_{1,\varepsilon/2}$ (resp. $V'_\varepsilon \subseteq U'_{\varepsilon/2}$) of $\hat{L}_1$ (resp. $\hat{L}'$) such that $V_{1,\varepsilon} \sqcup V'_\varepsilon$ is an $\hat{\mathcal{F}}$-saturated neighbourhood of $\hat{\mathcal{F}}(x)$. Since $\hat{\mathcal{F}}(x)$ is compact and $V_{1,\varepsilon} \sqcup V'_\varepsilon$ is an open neighbourhood of $\hat{\mathcal{F}}(x)$, there are finitely many connected components of $V_{1,\varepsilon} \sqcup V'_\varepsilon$ each of which intersects $\hat{\mathcal{F}}(x)$ and whose union is also a covering of $\hat{\mathcal{F}}(x)$. Let W_1 be the finite union in $V_{1,\varepsilon} \sqcup V'_\varepsilon$. Since $W_1 \supseteq \hat{\mathcal{F}}(x)$ and $g(C) \cap \hat{\mathcal{F}}(x) \neq \emptyset$ for any $g \in G$ and any connected component C of W_1, we have $G(W_1)$ also consists of finitely many connected components. Since $G(W_1) \subseteq \hat{\mathcal{F}}(W_1) \subseteq \overline{G(W_1)}$, we have that $\hat{\mathcal{F}}(W_1) \subseteq V_{1,\varepsilon} \sqcup V'_\varepsilon$ also consists of finitely many connected components. Let $W_{11}, \dots, W_{1l}$ be the connected components of $\hat{\mathcal{F}}(W_1)$ intersecting $V_{1,\varepsilon}$. Then $W := W_{11} \sqcup \cdots \sqcup W_{1l} \subset V_{1,\varepsilon}$ is a neighbourhood of $\hat{L}_1 = \hat{\mathcal{F}}_H(x)$ with $W_{1i} \cap \hat{L}_1 \neq \emptyset$ for any $i = 1, \dots, l$. We show that W is $\hat{\mathcal{F}}$-saturated. Indeed, since $\hat{\mathcal{F}}(W) \cap V_{1,\varepsilon} = W$ and $h(W_{1i}) \cap \hat{L}_1 \neq \emptyset$ for any $h \in H$ and $i = 1, \dots, l$, we have $V_{1,\varepsilon} \supseteq W = H(W)$. Since $\hat{\mathcal{F}}_H(W) \subseteq \overline{H(W)} \subseteq U_{1,\varepsilon}$, we have $\hat{\mathcal{F}}_H(W) \cap V'_\varepsilon = \emptyset$. Since $V_{1,\varepsilon} \sqcup V'_\varepsilon$ is an $\hat{\mathcal{F}}$-saturated neighbourhood of $\hat{\mathcal{F}}(W)$, we obtain $\hat{\mathcal{F}}_H(W) \subseteq V_{1,\varepsilon}$ and so $W = H(W) \subseteq \hat{\mathcal{F}}_H(W) \subseteq \hat{\mathcal{F}}(W) \cap V_{1,\varepsilon} = W$. Thus W is a desired $\hat{\mathcal{F}}_H$-saturated neighbourhood of $\hat{\mathcal{F}}_H(x)$ contained in $W \subseteq U_{1,\varepsilon} = B_\varepsilon(\hat{\mathcal{F}}_H(x))$. $\square$

4. Codimension one or two results

First, we consider the codimension one case.

Proposition 4.1. *Let G be an R-closed flow on a compact connected manifold X, H a finite index normal subgroup of G, and V the union of orbit closures of H*

which are codimension one connected elements. If there is a nonempty connected component of V which is open and consists of submanifolds, then $M/\hat{\mathcal{F}}_G$ is a closed interval or a circle such that there are at most two elements whose codimension more than one.

Note that dimensions in the above statement are Lebesgue covering dimensions.

Proof. By Lemma 3.2, we have that H is also R-closed. Put $\hat{\mathcal{F}} := \hat{\mathcal{F}}_H$. By Proposition 2.2, we have that $M/\hat{\mathcal{F}}$ is Hausdorff. Let U be the above open connected component of V. Put $p : M \to M/\hat{\mathcal{F}}$ the canonical projection. By Theorem 3.3 [D], we have that $p(U)$ is a 1-manifold. Since $\hat{\mathcal{F}}$ is R-closed, each connected component of the boundaries of $p(U)$ is a single point. Since U is open, we obtain that the boundaries $\partial U := \overline{U} - \mathrm{int} U$ have at least codimension two and so that $M - \partial U$ is connected. This implies that $M/\hat{\mathcal{F}}$ is a closed interval or a circle. Since G/H is a finite group acting $M/\hat{\mathcal{F}}$ and since a finite union of closures is a closure of finite union, we have that $M/\hat{\mathcal{F}}_G = (M/\hat{\mathcal{F}}_H)/(G/H)$ is either a closed interval or a circle and that there are at most two elements whose codimension more than one. □

Second, we consider the codimension two case. Consider the direct system $\{K_a\}$ of compact subsets of a topological space X and inclusion maps such that the interiors of K_a cover X. There is a corresponding inverse system $\{\pi_0(X - K_a)\}$, where $\pi_0(Y)$ denotes the set of connected components of a space Y. Then the set of ends of X is defined to be the inverse limit of this inverse system. By surfaces, we mean compact 2-dimensional manifolds with conners (i.e. locally modeled by $[0, 1]^2$).

Proposition 4.2. *Let G be an R-closed flow on a compact manifold X and H a finite index normal subgroup of G. Suppose that all orbit closures of H are closed connected submanifolds. If all but finitely many closures have codimension two and finite exceptions have codimension more than two, then $M/\hat{\mathcal{F}}_G$ is a surface.*

Proof. By Lemma 3.2, we have that H is also R-closed. Put $\hat{\mathcal{F}} := \hat{\mathcal{F}}_H$. Let $L_1, \ldots, L_k$ be all higher codimension elements of $\hat{\mathcal{F}}$. Removing higher codimensional elements, let M' be the resulting manifold, $\hat{\mathcal{F}}'$ the resulting decomposition of M'. Then $\hat{\mathcal{F}}'$ consists of codimension two closed connected submanifolds and is usc. By Theorem 3.12 [D3], we have that $M'/\hat{\mathcal{F}}'$ is a surface S'. Then $(M/\hat{\mathcal{F}}) - \{L_1, \ldots, L_k\} \cong M'/\hat{\mathcal{F}}' = S'$. We will show that S' has k ends. Indeed, since the exceptions L_i are finite, there are pairwise disjoint neighbourhoods U_i of them. Since $\hat{\mathcal{F}}$ is usc, there are pairwise disjoint saturated neighbourhoods $V_i \subseteq U_i$ of them. Since $W_i - L_i$ is connected for any connected neighbourhoods W_i of L_i, each end of S' is corresponded to one of L_i. This shows that S' has k ends corresponding to L_i. Since $M/\hat{\mathcal{F}}$ is compact metrizable, we have that $M/\hat{\mathcal{F}}$ is an end compactification of $M'/\hat{\mathcal{F}}'$ and so a surface S. Since G/H implies a finite group action on S and since any finite union of closures is the closure of finite union, we obtain that $M/\hat{\mathcal{F}}_G \cong (M/\hat{\mathcal{F}}_H)/(G/H)$ is a surface. □

5. Toral minimal sets

Recall that a vector field is trivial if it is identical or minimal. We obtain the following two statements for vector fields on 3-manifolds. For a flow v on a 3-manifold, a minimal set T of v is called a torusoid for v if there are an open

annulus A transverse to v and a circloid C in A whose saturation $\mathrm{Sat}_v(C)$ is T with $C = T \cap A$.

Lemma 5.1. *Let v be an R-closed vector field on a connected compact 3-manifold M. Then the union of torusoids is open and the quotient space is 1-dimensional.*

Proof. Let A be an annular transverse manifold of v and $C \subset A$ a circloid whose saturation is a torusoid T with $C = T \cap A$. Take small connected neighbourhoods $U_1, U_2 \subseteq \mathrm{Sat}_v(A)$ of T such that the time one map $f_v : U_1 \cap T \to U_2 \cap T$ is well defined. Since v is R-closed and so $\hat{\mathcal{F}}_v$ is usc, there is an $\hat{\mathcal{F}}_v$-saturated open neighbourhood $V \subseteq U_1 \cap U_2$ of T. Since T is $\hat{\mathcal{F}}_v$-saturated and connected, the connected component W of V is also an $\hat{\mathcal{F}}_v$-saturated open neighbourhood of T. Since $W \subseteq V$, we have that $W_A := W \cap A \subseteq U_1 \cap U_2 \cap A$ and so $f_{W_A} := f_v| : W_A \to W_A$ is well-defined. Since v is R-closed, the mapping f_{W_A} is a homeomorphism. Since $T \cap A = C \subset W_A$ is f_v invariant and U_1, U_2 are small, we have that W_A is connected. Let B be the union of the connected components of $A - W_A$ which are contractible in A. Then $B \cup W_A$ is an open annulus and $\mathrm{Sat}_v(B \cup W_A)$ is an $\hat{\mathcal{F}}_v$-saturated connected open neighbourhood of T which consists of torusoids. Indeed, define $\mathrm{Fill}_A(W_A)$ as follows: $p \in \mathrm{Fill}_A(W_A)$ if there is a simple closed curve in $W \cap A$ which bounds a disk in A containing $p \in A$. Since each point of B is bounded by a simple closed curve in W_A, we have that $B \sqcup W_A = \mathrm{Fill}_A(W_A)$ and $f' := f_v| : \mathrm{Fill}_A(W_A) \to \mathrm{Fill}_A(W_A)$ is a homeomorphism. Since C is a circloid and A is the annular neighbourhood, we obtain that $\mathrm{Fill}_A(W_A)$ is an open annulus. By the two point compactification of $\mathrm{Fill}_A(W_A)$, we obtain the resulting sphere S and the resulting homeomorphism f_S with the two fixed points which are the added new points. Since v is R-closed, we have that $M/\hat{v}$ is Hausdorff and so is $\mathrm{Fill}_A(W_A)/\hat{\mathcal{F}}_{f'}$. By the construction, $S/\hat{\mathcal{F}}_{f_S}$ is the two point compactification of $\mathrm{Fill}_A(W_A)/\hat{\mathcal{F}}_{f'}$. Since M is normal, we obtain that $S/\hat{\mathcal{F}}_{f_S}$ is Hausdorff and so f_S is R-closed. By Corollary 2.6 [Y2], we obtain that S consists of two fixed points and circloids. Then the open neighbourhood $\mathrm{Sat}_v(B \sqcup W_A) = \mathrm{Sat}_v(\mathrm{Fill}_A(W_A))$ of T consists of torusoids. Therefore the union of torusoids are open. □

Recall that a minimal set is trivial if a single orbit or the whole manifold.

Lemma 5.2. *Let v be an R-closed vector field on a connected compact 3-manifold M. Suppose that each two dimensional minimal set is torusoid. If there is a torusoid, then the orbit class space $M/\hat{v}$ of M is a closed interval or a circle.*

Proof. Note that each minimal set which is not a torusoid is a closed orbit. By Lemma 5.1, the union of torusoids is open and the quotient space of it is a 1-manifold. Since v is R-closed, the each boundary component of it is a single minimal set. Since the each boundary component is a closed orbit and so is codimension more than 1, the union of torusoids is connected. This implies that $M/\hat{v}$ is a closed interval or a circle. □

Lemma 5.3. *Let v be an R-closed vector field on a connected compact 3-manifold M. Suppose that each two dimensional minimal set is torusoid. If there are at least three distinct periodic orbits, then $\mathrm{Per}(v)$ is open dense and $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$.*

Proof. Let $p : M \to M/\hat{v}$ be the canonical projection. By Lemma 5.2, there are no two dimensional orbits closures. Therefore $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$. Since $\mathrm{Sing}(v)$ is closed, we have that $\mathrm{Per}(v)$ is open. On the other hand, by Theorem

3.12 [D3], the quotient space of $\mathrm{Per}(v)$ is a 2-dimensional manifold with conner. Since $M/\hat{v}$ is compact metrizable, by Urysohn's theorem, the Lebesgue covering dimension and the small inductive dimension are corresponded in $M/\hat{v}$. Hence the boundary $\partial(p(\mathrm{Per}(v)))$ of $p(\mathrm{Per}(v))$ has at most the small inductive dimension one. Since $\partial(p(\mathrm{Per}(v)))$ consists of singularities, we obtain that $\partial(\mathrm{Per}(v))$ has at most the small inductive dimension one and so the Lebesgue covering dimension one. Therefore $M - \partial(\mathrm{Per}(v))$ is connected and so $M - \partial(\mathrm{Per}(v)) = \mathrm{Per}(v)$. This shows that $\mathrm{Per}(v)$ is dense. □

Now, we state the following trichotomy that v is either "almost one dimensional" or "almost two dimensional" or with "complicated" minimal sets.

Proposition 5.4. *Let v be an nontrivial R-closed vector field on a connected compact 3-manifold M. Then one of the following holds:*
1) The orbit class space $M/\hat{v}$ of M is a closed interval or a circle and each interior point of the orbit class is two dimensional.
2) $\mathrm{Per}(v)$ is open dense and $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$.
3) There is a nontrivial minimal set which is not a torusoid.

Proof. Suppose that each nontrivial minimal set is a torusoid. Since v is nontrivial, there is a minimal set. If the minimal set is two dimensional, then Lemma 5.2 implies that 1). Otherwise we may assume that there are no two dimensional minimal sets. Since v is nontrivial, there is a periodic orbit. By flow box theorem, this has a neighbourhood without singularities. Thus there are infinitely many periodic orbits. By Lemma 5.3, we have that 2) holds. □

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